JURONG JUNIOR COLLEGE

J2 Preliminary Examination

MATHEMATICS
Higher 2

9740/01

18 August 2009

Paper 1

3 hours

Additional materials: Answer Paper
List of Formulae (MF15)
Cover Page

READ THESE INSTRUCTIONS FIRST

Write your name and civics class on all the work you hand in.
Write in dark blue or black pen on both sides of the paper.
You may use a soft pencil for any diagrams or graphs.
Do not use staples, paper clips, highlighters, glue or correction fluid.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use a graphic calculator.
Unsupported answers from a graphic calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphic calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.

The number of marks is given in brackets [ ] at the end of each question or part question.
At the end of the examination, fasten all your work securely together, with the cover page in front.

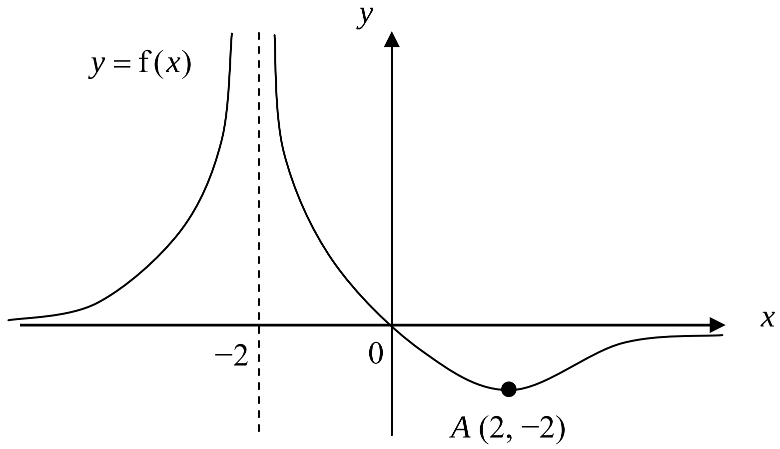
This document consists of **5** printed pages and **1** blank page.

1 The diagram below shows the graph of $y = \mathrm{f}(x)$. Sketch, on separate diagrams, the following graphs, indicating clearly any asymptotes, axial intercepts and turning points, where possible.

(i) $y = -\mathrm{f}(x) - 2$, [2]

(ii) $y = \mathrm{f}(|x|)$, [1]

(iii) $y = \mathrm{f}'(x)$. [3]



2 The functions f and g are defined to be

$$\mathrm{f} : x \mapsto 4 - 2e^x, \quad x \in \Re,\ x \le 0 \quad \text{and} \quad \mathrm{g} : x \mapsto 1 + \frac{1}{x-1}, \quad x \in \Re,\ x \ge 0, x \ne 1.$$

(i) Sketch the graph of f and state its range. [2]

(ii) Prove that the composite function gf exists and define it giving its rule and domain. [3]

(iii) Sketch the graph of g and hence find the range of gf. [2]

3 On the same Argand diagram, sketch the loci

(i) $\left|z - 2\sqrt{3} - 2i\right| = 2$, [2]

(ii) $z - z^* = 8i$, [2]

(iii) $\arg\left(z - \sqrt{3} - i\right) = \dfrac{\pi}{3}$. [2]

The complex number w is represented by the point of intersection of the loci in parts (i), (ii) and (iii). Find w, in the form $x + iy$, giving the exact values of x and y. [1]

4 A student has \$1000 in his savings account initially. He decides to deposit \$5 from his pocket money each week into the account, which pays a compound interest rate of 1% per year. Taking a year to consist of 52 weeks and P to be the amount of money in the account after t years, show that the differential equation

$$\frac{\mathrm{d}P}{\mathrm{d}t} = 0.01P + 260$$

may be used to model the balance in the student's bank account. [1]

(i) Express P in terms of t . [4]

(ii) How long will it take for the balance to exceed \$2000 ? [2]

5 **(a)** Find $\int \frac{1-4x}{4x^2+1}\,\mathrm{d}x$. [3]

(b) By means of the substitution $x = \tan\theta$, find the exact value of

$$\int_1^\infty \frac{x^2}{\left(x^2+1\right)^2}\,\mathrm{d}x .$$ [5]

6 **(a)** On a farm there are 3 different types of animals : chickens, horses, and sheep. The farmer confirms that the number of sheep is twice the number of chickens. He also counted a total of 1250 animal legs. Due to his handwriting, he was not sure if there were 250 animals or 350 animals in total.

Find the correct number of chickens, horses and sheep. [4]

(b) By using an algebraic method, solve the inequality $\frac{2x-18}{x+3} > x-6$. [3]

Hence, solve the inequality $\frac{\ln x^2 - 18}{\ln x + 3} > \ln x - 6$. [3]

7 **(a)** A sequence of integers $u_0, u_1, u_2, \cdots$ is defined by $u_0 = 1$ and $u_{n+1} = 3u_n - 7$.

Prove by induction that $u_n = \frac{1}{2}\left[7 - 5(3)^n\right]$ for all non-negative integral values of n. [4]

(b) Prove that for all positive integers n, $\frac{1}{n!} - \frac{1}{(n+1)!} = \frac{1}{n! + (n-1)!}$. [2]

Hence evaluate $\sum_{n=1}^{N} \frac{1}{n! + (n-1)!}$ in terms of N. [2]

Deduce that $\sum_{n=1}^{\infty} \frac{1}{n!} < 2$. [2]

8 A geometric series has first term a, common ratio r and nth term denoted by G_n. An arithmetic series has first term a, common difference d and nth term denoted by A_n. It is given that a, r and d are non-zero and the two series are related by the following equations

$$A_4 - A_2 = G_3 \quad \text{and} \quad 5A_5 - 6a = 9G_5.$$

Show that $9r^4 - 10r^2 + 1 = 0$. [4]

It is also given that $r > 0$, $r \neq 1$.

(i) Deduce that the geometric series is convergent and show that its sum to infinity is $\frac{3}{2}a$. [3]

(ii) Find the least value of N for which $\sum_{n=1}^{N} A_n > 10 \sum_{n=1}^{\infty} G_n$ where $a > 0$. [4]

9 Relative to a fixed point O, the position vectors of the points A, B and C are given as follows :

$$\overrightarrow{OA} = \mathbf{i} + \mathbf{k}, \quad \overrightarrow{OB} = 2\mathbf{i} + \mathbf{j}, \quad \overrightarrow{OC} = 4\mathbf{i} + 2\mathbf{j} + \mathbf{k}.$$

It is known that A, B and C exist in the plane Π_1.

(i) Show that the equation of plane Π_1 is $2x - 3y - z = 1$. [2]

(ii) The point D has position vector $5\mathbf{i} + \mathbf{j} - \mathbf{k}$. Find the position vector of the foot of perpendicular from D to plane Π_1 and hence find the exact distance from D to plane Π_1. [4]

(iii) It is further known that the point D is the reflection of point C about another plane Π_2. Show that the equation of plane Π_2 is $x - y - 2z = 3$.

Find the equation of the line of intersection between planes Π_1 and Π_2. [5]

10 **(a)** Given that

$$\mathrm{i}(w+2) = z \quad \text{and} \quad wz = -4\mathrm{i}, \quad \text{where } \mathrm{Im}(w) < 0.$$

Find exact values of z and w in the form $a + \mathrm{i}b$, where a and b are real. [3]

(b) Find the fourth roots of $-16\mathrm{i}$, giving your answers exactly, in the form $re^{\mathrm{i}\theta}$, where $-\pi < \theta \le \pi$. [3]

(c) Express $z = 1 - \sqrt{3}\,\mathrm{i}$ in modulus-argument form. [2]

Hence, find the set of positive integral values of n for which z^n is real and negative. [3]

11 **(a)** A curve is defined by the parametric equations

$$x = 3t, \quad y = \frac{3}{t}$$

The point P on the curve has parameter $t = 2$. Show that the normal at P has equation $2y = 8x - 45$. The normal meets the curve again at the point Q. Find the value of t at Q. [6]

(b) The volume V and surface area A of a closed cylinder of radius r and height h are increasing at a rate of 15 m^3/s and 30 m^2/s respectively. Given that its height remains constant, obtain an expression for r in terms of h.

Deduce that $h > 1$. [6]