Raffles Institution
H2 Mathematics
Solution for 2014 A-Level Paper 1

Question 1

No.	Suggested Solution	Remarks for Student
(i)	Let $y=\mathrm{f}\left(x\right)=\dfrac{1}{1-x}$ $1-x=\dfrac{1}{y}\Rightarrow x=1-\dfrac{1}{y}=\dfrac{y-1}{y}$ $\therefore \mathrm{f}^{-1}\left(x\right)=\dfrac{x-1}{x}$ $\mathrm{f}^2\left(x\right)=\mathrm{f}\left(\dfrac{1}{1-x}\right)=\dfrac{1}{1-\dfrac{1}{1-x}}=\dfrac{1}{\dfrac{1-x-1}{1-x}}=\dfrac{x-1}{x}=\mathrm{f}^{-1}\left(x\right)$	
(ii)	$\mathrm{f}^3\left(x\right)=\mathrm{f}\left(\mathrm{f}^2\left(x\right)\right)=\mathrm{f}\left(\mathrm{f}^{-1}\left(x\right)\right)=x$	

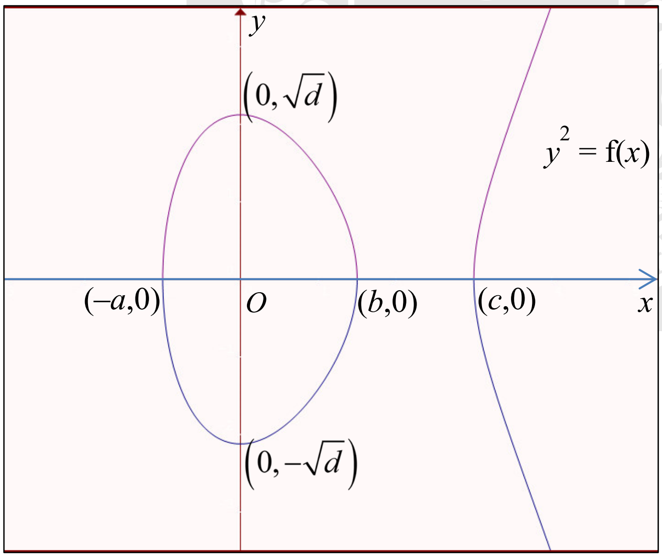
Question 2

No.	Suggested Solution	Remarks for Student
2	$x^2y+xy^2+54=0\ \ldots\ (1)$ Differentiate with respect to x: $x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}+2xy+2xy\dfrac{\mathrm{d}y}{\mathrm{d}x}+y^2=0$ When $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1$, we have $-x^2+2xy-2xy+y^2=0$ $y^2=x^2\Rightarrow y=\pm x$ Sub $y=x$ into (1): $x^3+x^3+54=0\Rightarrow x^3=-27\Rightarrow x=-3=y$ Sub $y=-x$ into (1): $-x^3+x^3+54=0\Rightarrow 54=0$ (no solution) Therefore there is only one point on C with gradient -1, and coordinates of that point is $\left(-3,-3\right)$.	It is not necessary to make $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ the subject before substituting $\dfrac{\mathrm{d}y}{\mathrm{d}x}=-1$.

Question 3

No.	Suggested Solution	Remarks for Student
(i)	$\mathbf{a} \times \mathbf{b} = \mathbf{0}$ Either $\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ or $\mathbf{a}$ and $\mathbf{b}$ are parallel to each other.	$\mathbf{a} = \mathbf{0}$ or $\mathbf{b} = \mathbf{0}$ are trivial cases where students might discard.
(ii)	$\mathbf{n} = \dfrac{1}{\sqrt{1^2 + 2^2 + (-2)^2}} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} - \dfrac{2}{3}\mathbf{k}$ Alternative answer: $\mathbf{n} = -\left(\dfrac{1}{3}\mathbf{i} + \dfrac{2}{3}\mathbf{j} - \dfrac{2}{3}\mathbf{k}\right)$	Read question carefully and note that they asked for unit vector.
(iii)	cosine of acute angle $= \dfrac{1}{\sqrt{1^2 + 2^2 + (-2)^2}} \left\| \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\| = \dfrac{2}{3}$	

Question 4

No.	Suggested Solution	Remarks for Student
(i)		
(ii)	They are the vertical lines $x = -a$, $x = b$ and $x = c$. Alternative: Tangents at these points are parallel to the y axis.	Not sufficient to say "tangents are undefined"

Question 5

No.	Suggested Solution	Remarks for Student
(i)	$z=1+2\mathrm{i}$ $z^2=(1+2\mathrm{i})^2=1+4\mathrm{i}-4=-3+4\mathrm{i}$ $z^3=(-3+4\mathrm{i})(1+2\mathrm{i})=-3+4\mathrm{i}-6\mathrm{i}-8=-11-2\mathrm{i}$ $\dfrac{1}{z^3}=\dfrac{1}{-11-2\mathrm{i}}=\dfrac{1}{125}(-11+2\mathrm{i})=-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}$	Read question carefully, we need $\dfrac{1}{z^3}$, not just z^3
(ii)	$pz^2+\dfrac{q}{z^3}=p(-3+4\mathrm{i})+q\left(-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}\right)$ is real Therefore, $4p+\dfrac{2}{125}q=0\Rightarrow q=-250p$ $pz^2+\dfrac{q}{z^3}=p(-3+4\mathrm{i})-250p\left(-\dfrac{11}{125}+\dfrac{2}{125}\mathrm{i}\right)$ $=-3p+4p\mathrm{i}+22p-4p\mathrm{i}$ $=19p$	Note that $pz^2+\dfrac{q}{z^3}$ is real. This is a "hint" to "check" your answer.

Question 6

No.	Suggested Solution	Remarks for Student
(a)(i)	Let P_n be the statement $p_n=\dfrac{1}{3}(7-4^n)$ for $n\in\mathbb{Z}^+$. When $n=1$, LHS $=p_1=1$ RHS $=\dfrac{1}{3}(7-4^1)=\dfrac{1}{3}(3)=1$ LHS = RHS, so P_1 is true. Now assume that P_k is true for some $k\in\mathbb{Z}^+$, i.e. $p_k=\dfrac{1}{3}(7-4^k)$ We need to show that P_{k+1} is true, i.e. $p_{k+1}=\dfrac{1}{3}(7-4^{k+1})$ We have $p_{k+1}=4p_k-7$ $=4\left[\dfrac{1}{3}(7-4^k)\right]-7$	

	$=\frac{4}{3}\left(7-4^{k}\right)-7$ $=\frac{1}{3}\left(28-4^{k+1}-21\right)$ $=\frac{1}{3}\left(7-4^{k+1}\right)$ Thus P_k is true $\Rightarrow P_{k+1}$ is true. Since P_1 is also true, P_n is true for all $n\in\mathbb{Z}^+$ by Mathematical Induction.	
(a)(ii)	$\sum_{r=1}^{n}p_r=\frac{1}{3}\sum_{r=1}^{n}\left(7-4^r\right)$ $=\frac{1}{3}\left(7n-\sum_{r=1}^{n}4^r\right)$ $=\frac{1}{3}\left(7n-\frac{4\left(4^n-1\right)}{4-1}\right)$ $=\frac{7n}{3}-\frac{4}{9}\left(4^n-1\right)$	
(b)(i)	$\sum_{r=1}^{n}u_r=S_n=1-\frac{1}{(n+1)!}$ As $n\to\infty$, $\frac{1}{(n+1)!}\to 0$. Therefore, $\sum_{r=1}^{n}u_r$ converges. $\sum_{r=1}^{\infty}u_r=1$.	
(b)(ii)	For $n\geq 2$, $u_n=S_n-S_{n-1}$ $=1-\frac{1}{(n+1)!}-\left(1-\frac{1}{n!}\right)$ $=\frac{1}{n!}-\frac{1}{(n+1)!}$ $=\frac{(n+1)-1}{(n+1)!}$ $=\frac{n}{(n+1)!}$ $u_1=S_1=1-\frac{1}{(1+1)!}=\frac{1}{2}=\frac{1}{(1+1)!}$ $\therefore u_n=\frac{n}{(n+1)!}$ for $n\geq 1$.	

Question 7

No.	Suggested Solution	Remarks for Student
(i)	$\alpha = 1.88525 = 1.885$ (3 d.p.) At $(\beta, -7)$, $\beta^6 - 3\beta^4 - 7 = -7$ $\beta^4(\beta^2 - 3) = 0$ $\beta = 0$ or $\sqrt{3}$ or $-\sqrt{3}$. From diagram, $\beta > 0$, thus $\beta = \sqrt{3}$	
(ii)	$\int_\beta^\alpha \mathrm{f}(x)\,\mathrm{d}x = -0.597$ (3 d.p.)	
(iii)	$\int_0^{\sqrt{3}} \left(x^6 - 3x^4 - 7\right)\mathrm{d}x = \left[\frac{1}{7}x^7 - \frac{3}{5}x^5 - 7x\right]_0^{\sqrt{3}}$ $= \frac{1}{7}\left(\sqrt{3}\right)^7 - \frac{3}{5}\left(\sqrt{3}\right)^5 - 7\sqrt{3}$ $= \frac{27}{7}\sqrt{3} - \frac{27}{5}\sqrt{3} - 7\sqrt{3}$ $= -\frac{299}{35}\sqrt{3}$ Required area $= \frac{299}{35}\sqrt{3} - 7\sqrt{3} = \frac{54}{35}\sqrt{3}$	
(iv)	$\mathrm{f}(x) = x^6 - 3x^4 - 7 = (-x)^6 - 3(-x)^4 - 7 = \mathrm{f}(-x)$ (Shown) Thus, f is symmetrical about the y-axis. Both α and $-\alpha$ are the real roots of f(x) = 0. The other 4 roots are non-real. Since the coefficients and constant term of f are real, the 4 roots will be 2 conjugate pairs.	Question asked students to comment on the 6 roots. Thus concepts of complex numbers are required for (iv)

Question 8

No.	Suggested Solution	Remarks for Student
(i)	$\int \mathrm{f}(x)\,\mathrm{d}x = \int \frac{1}{\sqrt{9 - x^2}}\,\mathrm{d}x = \sin^{-1}\frac{x}{3} + c$	

(ii)	$\begin{aligned} \mathrm{f}(x) &= \frac{1}{\sqrt{9-x^2}} \\ &= \frac{1}{3\sqrt{1-\frac{x^2}{9}}} \\ &= \frac{1}{3}\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}} \\ &\approx \frac{1}{3}\left(1+\left(-\frac{1}{2}\right)\left(-\frac{x^2}{9}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2}\left(-\frac{x^2}{9}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{6}\left(-\frac{x^2}{9}\right)^3\right) \\ &= \frac{1}{3}\left(1+\frac{x^2}{18}+\frac{3x^4}{648}+\frac{5x^6}{11664}\right) \\ &= \frac{1}{3}+\frac{x^2}{54}+\frac{x^4}{648}+\frac{5x^6}{34992} \end{aligned}$	As stated in the question, the intended method is to apply Binomial Theorem on $\left(1-\frac{x^2}{9}\right)^{-\frac{1}{2}}$ instead of finding the Maclaurin's expansion by repeated differentiation.
(iii)	$\begin{aligned} \sin^{-1}\frac{x}{3}+c &= \int \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x \\ &\approx \int\left(\frac{1}{3}+\frac{x^2}{54}+\frac{x^4}{648}+\frac{5x^6}{34992}\right)\mathrm{d}x \\ &= \frac{1}{3}x+\frac{x^3}{162}+\frac{x^5}{3240}+\frac{5x^7}{244944} \end{aligned}$ Note that $c=0$ when $x=0$. Thus, $\sin^{-1}\frac{x}{3} \approx \frac{1}{3}x+\frac{x^3}{162}+\frac{x^5}{3240}+\frac{5x^7}{244944}$	

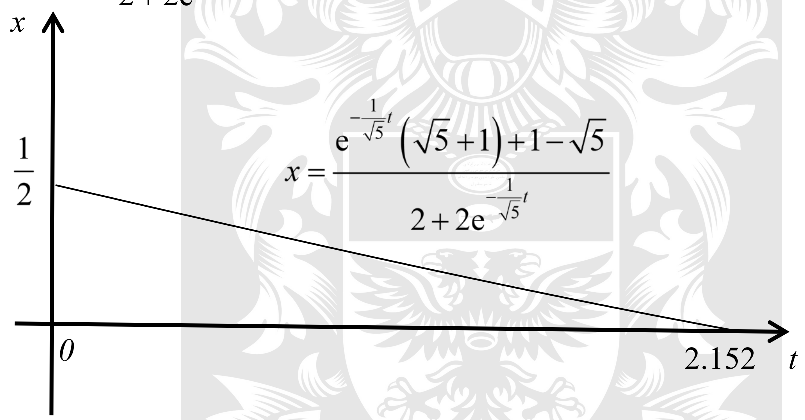
Question 9

No.	Suggested Solution	Remarks for Student
	$p:\mathbf{r}\cdot\begin{pmatrix}1\\2\\-3\end{pmatrix}=12, \qquad l:\mathbf{r}=\begin{pmatrix}1\\-1\\3\end{pmatrix}+\lambda\begin{pmatrix}2\\-1\\4\end{pmatrix},\ \lambda\in\mathbb{R}$	
(i)	$\begin{pmatrix}1\\2\\-3\end{pmatrix}\times\begin{pmatrix}2\\-1\\4\end{pmatrix}=\begin{pmatrix}5\\-10\\-5\end{pmatrix}=-5\begin{pmatrix}-1\\2\\1\end{pmatrix}$ $q:\mathbf{r}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}1\\-1\\3\end{pmatrix}\cdot\begin{pmatrix}-1\\2\\1\end{pmatrix}=-1-2+3=0$ Therefore the Cartesian equation of q is $-x+2y+z=0$	
(ii)	Using GC, $m:\mathbf{r}=\begin{pmatrix}6\\3\\0\end{pmatrix}+\mu\begin{pmatrix}4\\1\\2\end{pmatrix},\ \mu\in\mathbb{R}$	
(iii)	$\overrightarrow{OB}=\begin{pmatrix}6+4\mu\\3+\mu\\2\mu\end{pmatrix}$ $\overrightarrow{AB}=\begin{pmatrix}6+4\mu-1\\3+\mu+1\\2\mu-3\end{pmatrix}=\begin{pmatrix}5+4\mu\\4+\mu\\2\mu-3\end{pmatrix}$ Let $y=\left\|\overrightarrow{AB}\right\|^2=(5+4\mu)^2+(4+\mu)^2+(2\mu-3)^2$ $\frac{\mathrm{d}y}{\mathrm{d}\mu}=8(5+4\mu)+2(4+\mu)+4(2\mu-3)=0$ $\Rightarrow 40+32\mu+8+2\mu+8\mu-12=0$ $\Rightarrow 42\mu=-36$ $\therefore \mu=-\frac{6}{7}$ $\frac{\mathrm{d}^2y}{\mathrm{d}\mu^2}=32+2+8=42>0 \ \Rightarrow \mu=-\frac{6}{7}$ gives the min value of y. Coordinates of required point $=\left(6+4\left(-\frac{6}{7}\right),3-\frac{6}{7},2\left(-\frac{6}{7}\right)\right)=\left(\frac{18}{7},\frac{15}{7},-\frac{12}{7}\right)$	

Question 10

No.	Suggested Solution	Remarks for Student
	$\frac{\mathrm{d}x}{\mathrm{d}t}=k\left(1+x-x^2\right)$	
(i)	$x=\frac{1}{2}$, $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{4}$ when $t=0$, we have $-\frac{1}{4}=k\left(1+\frac{1}{2}-\frac{1}{4}\right)=\frac{5}{4}k\Rightarrow k=-\frac{1}{5}$ (Shown)	
(ii)	$1+x-x^2 \quad =-\left(x^2-x\right)+1$ $=-\left(x^2-x+\frac{1}{4}-\frac{1}{4}\right)+1$ $=\frac{5}{4}-\left(x-\frac{1}{2}\right)^2$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{5}\left(\frac{5}{4}-\left(x-\frac{1}{2}\right)^2\right) \quad \Rightarrow\int\frac{1}{\frac{5}{4}-\left(x-\frac{1}{2}\right)^2}\,\mathrm{d}x=\int-\frac{1}{5}\,\mathrm{d}t$ $-\frac{1}{5}t \;=\frac{1}{2\left(\frac{\sqrt{5}}{2}\right)}\ln\left\lvert\frac{\frac{\sqrt{5}}{2}+\left(x-\frac{1}{2}\right)}{\frac{\sqrt{5}}{2}-\left(x-\frac{1}{2}\right)}\right\rvert+c$ $=\frac{1}{\sqrt{5}}\ln\left(\frac{\frac{\sqrt{5}}{2}+x-\frac{1}{2}}{\frac{\sqrt{5}}{2}-x+\frac{1}{2}}\right)+c$, since $0\le x\le\frac{1}{2}\Leftrightarrow-\frac{1}{2}\le x-\frac{1}{2}\le 0$ Thus, $t=-\sqrt{5}\ln\left(\frac{2x+\sqrt{5}-1}{-2x+\sqrt{5}+1}\right)+c$ $x=\frac{1}{2}$, $\frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{1}{4}$ when $t=0$, we have $0=-\sqrt{5}\ln\left(\frac{1+\sqrt{5}-1}{-1+\sqrt{5}+1}\right)+c\Rightarrow c=\sqrt{5}\ln\left(1\right)=0$ $\therefore t=-\sqrt{5}\ln\left(\frac{2x+\sqrt{5}-1}{-2x+\sqrt{5}+1}\right)$	

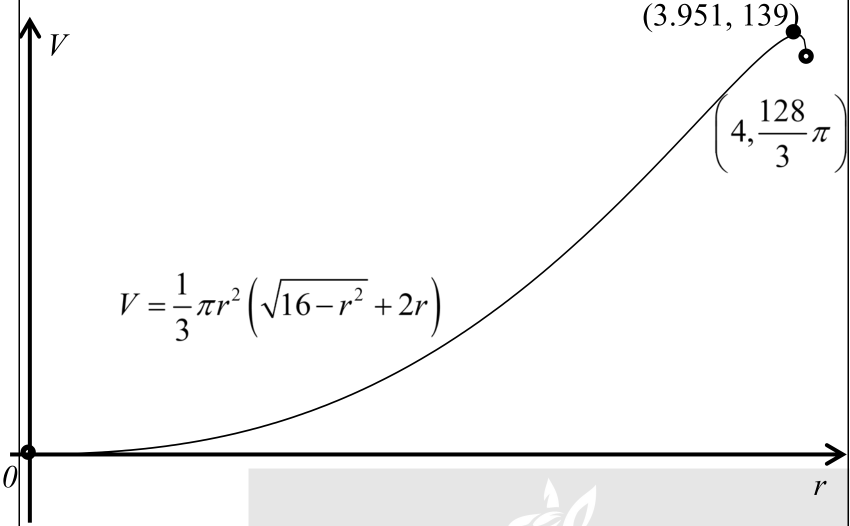
AUSPICIUM MELIORIS AEVI

(iii) (a)	$x=\frac{1}{4}\Rightarrow t=-\sqrt{5}\ln\left(\frac{\frac{1}{2}+\sqrt{5}-1}{-\frac{1}{2}+\sqrt{5}+1}\right)=\sqrt{5}\ln\left(\frac{2\sqrt{5}+1}{2\sqrt{5}-1}\right)$	
(iii) (b)	$x=0\Rightarrow t=-\sqrt{5}\ln\left(\frac{\sqrt{5}-1}{\sqrt{5}+1}\right)=2.152\text{ min (3 d.p.)}$	
(iv)	$t=-\sqrt{5}\ln\left(\frac{2x+\sqrt{5}-1}{-2x+\sqrt{5}+1}\right)$ $e^{-\frac{1}{\sqrt{5}}t}=\frac{2x+\sqrt{5}-1}{-2x+\sqrt{5}+1}$ $e^{-\frac{1}{\sqrt{5}}t}\left(-2x+\sqrt{5}+1\right)=2x+\sqrt{5}-1$ $x\left(2+2e^{-\frac{1}{\sqrt{5}}t}\right)=e^{-\frac{1}{\sqrt{5}}t}\left(\sqrt{5}+1\right)+1-\sqrt{5}$ $\therefore x=\frac{e^{-\frac{1}{\sqrt{5}}t}\left(\sqrt{5}+1\right)+1-\sqrt{5}}{2+2e^{-\frac{1}{\sqrt{5}}t}}$  	

AUSPICIUM MELIORIS AEVI

Question 11

No.	Suggested Solution	Remarks for Student
(i)	$h^2+r^2=4^2 \Rightarrow h=\sqrt{16-r^2}$ $V \quad =\frac{1}{3}\pi r^2 h+\frac{1}{2}\left(\frac{4}{3}\pi r^3\right)$ $\quad =\frac{1}{3}\pi r^2\sqrt{16-r^2}+\frac{2}{3}\pi r^3$ $\quad =\frac{1}{3}\pi r^2\left(\sqrt{16-r^2}+2r\right)$ $\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{2}{3}\pi r\left(\sqrt{16-r^2}+2r\right)+\frac{1}{3}\pi r^2\left(\frac{1}{2\sqrt{16-r^2}}(-2r)+2\right)$ When $r=r_1$, $\frac{\mathrm{d}V}{\mathrm{d}r}=0$ $\frac{2}{3}\pi r\left(\sqrt{16-r^2}+2r\right)+\frac{1}{3}\pi r^2\left(\frac{1}{2\sqrt{16-r^2}}(-2r)+2\right)=0$ $2\sqrt{16-r^2}+4r-\frac{r^2}{\sqrt{16-r^2}}+2r=0$ $2\left(16-r^2\right)+6r\left(\sqrt{16-r^2}\right)-r^2=0$ $6r\left(\sqrt{16-r^2}\right)=3r^2-32$ $36r^2\left(16-r^2\right)=9r^4-192r^2+1024$ $576r^2-36r^4=9r^4-192r^2+1024$ $45r^4-768r^2+1024=0$ (Shown)	
(ii)	$45r^4-768r^2+1024=0$ From GC, $r \approx 1.20742$ or ≈ 3.950797 $=1.207$ (3dp) or 3.951 (3dp)	
(iii)	$\frac{\mathrm{d}V}{\mathrm{d}r}=\frac{2}{3}\pi r\left(\sqrt{16-r^2}+2r\right)+\frac{1}{3}\pi r^2\left(\frac{1}{2\sqrt{16-r^2}}(-2r)+2\right)$ Using GC, $\left.\frac{\mathrm{d}V}{\mathrm{d}r}\right\|_{r=1.20742}=18.311\neq 0$, $\Rightarrow r=1.20742$ does not give a stationary value of V. Thus, $r_1=3.951$ and corresponding $h=\sqrt{16-r_1^2}=0.625$ (3sf)	

(iv)		Note that the curve is only valid for $0<r<4$.